

# EIGENVALUE ESTIMATES FOR THE DIRAC-SCHRÖDINGER OPERATORS

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ABSTRACT. We give new estimates for the eigenvalues of the hypersurface Dirac operator in terms of the intrinsic energy-momentum tensor, the mean curvature and the scalar curvature. We also discuss their limiting cases as well as the limiting cases of the estimates obtained by X. Zhang and O. Hijazi in [12] and [8]. We compare these limiting cases with those corresponding to the Friedrich and Hijazi inequalities. We conclude by comparing these results to intrinsic estimates for the Dirac-Schrödinger operator  $D_f = D - \frac{f}{2}$ .

## 1. INTRODUCTION

In this paper, we start by comparing the hypersurface spinor bundle  $S$  of a hypersurface  $M$  to the fundamental spinor bundle  $\Sigma M$  of  $M$ . The hypersurface spinor bundle  $S$  is obtained by restricting the spinor bundle of the ambient space  $N$  to  $M$ . If  $\varphi \in \Gamma(S)$  is a section of this bundle, the energy-momentum tensor  $Q^\varphi$  associated with  $\varphi$  is defined on the complement of its zero set, by

$$Q_{ij}^\varphi = \frac{1}{2}(e_i \cdot \nu \cdot \nabla_j \varphi + e_j \cdot \nu \cdot \nabla_i \varphi, \varphi / |\varphi|^2),$$

where  $\nu$  is a unit normal vector field globally defined along  $M$ ,  $e_i, e_j$  are vectors of a local orthonormal frame of  $M$ , and where  $\nabla_i \varphi$  stands for the covariant derivative of the spinor field  $\varphi$  in the direction of  $e_i$ . Then the Schrödinger-Lichnerowicz formula for the classical Dirac operator  $D$  on  $M$  leads to the following result (compare with [12]):

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**Theorem 1.1.** *Let  $M^n \subset (N^{n+1}, \tilde{g})$  be a compact hypersurface of a Riemannian spin manifold  $N$ . Let  $\lambda$  be any eigenvalue of the hypersurface Dirac operator  $D_H = D - \frac{H}{2}$ , associated with an eigenspinor  $\varphi$ . Assume that  $R + 4|Q^\varphi|^2 > H^2 > 0$ , then one has*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{R + 4|Q^\varphi|^2} - |H| \right)^2. \quad (1)$$

where  $R$  and  $H$  are respectively the scalar curvature and the mean curvature of  $M$ , and  $Q^\varphi$  is the energy-momentum tensor associated with  $\varphi$ .

In fact, we see that if  $M$  is a minimal hypersurface, the hypersurface Dirac operator corresponds to the classical Dirac operator. Therefore, in this case, this estimate is exactly the one given by O. Hijazi in [7].

We then discuss the limiting case of equation (1) and that given by X. Zhang in [12].

As in [6] and [8], we then prove

**Theorem 1.2.** *Under the same conditions as in Theorem 1.1, suppose that  $\bar{R} e^{2u} + 4|Q^\varphi|^2 > H^2 > 0$ , where  $\bar{R}$  is the scalar curvature of  $M$  for some conformal metric  $\bar{g} = e^{2u} \tilde{g}$ , with  $du(\nu)|_M \equiv 0$ , then*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{\bar{R} e^{2u} + 4|Q^\varphi|^2} - |H| \right)^2. \quad (2)$$

The discussion of the limiting case in this inequality and that proved in [8] is similar to that of (1). As a conclusion, we observe that these inequalities correspond to a generalization of the classical estimates in terms of the Dirac-Schrödinger operator  $D_f = D - \frac{f}{2}$ , for a real function  $f$  on  $M$ .

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## 2. PRELIMINARIES

**2.1. Restriction of Spinors to the Hypersurface.** In this paper we will consider an oriented compact hypersurface  $(M^n, g)$  of a Riemannian spin manifold  $(N^{n+1}, \tilde{g})$ , with a spin structure  $\text{Spin}N$ . The metric  $g$  is the induced metric

on  $M$  by  $\tilde{g}$ . The possibility to define globally a unit normal vector field  $\nu$  on  $M$  allows to induce from  $\text{Spin}N$  a spin structure on  $M$ , denoted by  $\text{Spin}M$ . For this, we can associate to every oriented orthonormal frame  $(e_1, \dots, e_n)$  on  $M$  an oriented orthonormal frame  $(e_1, \dots, e_n, \nu)$  of  $N$  such that the principal  $\text{SO}(n)$ -bundle  $\text{SO}_n M$  of oriented orthonormal frames on  $M$  is identified with a sub-bundle of  $\text{SO}_{n+1} N|_M$ . Such a map is denoted by  $\Phi$ .

Let  $\mathbb{C}l_n$  be the  $n$ -dimensional complex Clifford algebra and  $\mathbb{C}l_n^0$  its even part. Recall that there exists an isomorphism

$$\begin{aligned} \alpha : \mathbb{C}l_n &\longrightarrow \mathbb{C}l_{n+1}^0 \\ e_i &\longmapsto e_i \cdot \nu. \end{aligned} \tag{3}$$

Here,  $\nu$  stands for the last vector of the canonical basis of  $\mathbb{R}^{n+1}$ .

In particular,  $\alpha$  yields the following commutative diagram:

$$\begin{array}{ccc} \text{Spin}(n) & \xrightarrow{\alpha} & \text{Spin}(n+1) , \\ \downarrow \text{Ad} & & \downarrow \text{Ad} \\ \text{SO}(n) & \hookrightarrow & \text{SO}(n+1) \end{array}$$

where the inclusion of  $\text{SO}(n)$  in  $\text{SO}(n+1)$  is that which fixes the last basis vector under the action of  $\text{SO}(n+1)$  on  $\mathbb{R}^{n+1}$ , and  $\text{Ad}$  the adjoint representation of  $\text{Spin}(n)$  on  $\text{SO}(n)$ , which is given by

$$\text{Ad}_\eta(x) = \eta \cdot x \cdot \eta^{-1}$$

for all  $\eta \in \text{Spin}(n)$  and  $x \in \mathbb{R}^n$ .

This allows to pull back via  $\Phi$  the fiber bundle  $\text{Spin}N|_M$  on  $\text{SOM}$  as a spin structure for  $M$ , denoted by  $\text{Spin}M$ . The projection of  $\text{Spin}M$  on  $\text{SOM}$ , as well as the projection of  $\text{Spin}N$  on  $\text{SON}$ , is denoted by  $\pi$ . Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Spin}M & \xrightarrow{\Phi^*} & \text{Spin}N|_M , \\ \downarrow \pi & & \downarrow \pi \\ \text{SOM} & \xrightarrow{\Phi} & \text{SON}|_M \end{array}$$

Let  $\Sigma N$  be the spinor bundle on  $N$ , i.e.,

$$\Sigma N = \text{Spin}N \times_{\rho_{n+1}} \Sigma_{n+1},$$

where  $\rho_{n+1}$  is the restriction to  $\text{Spin}(n+1)$  of an irreducible representation of  $\mathbb{C}l_{n+1}$  on the space of spinors  $\Sigma_{n+1}$ , of dimension  $2^{\lfloor \frac{n+1}{2} \rfloor}$  ( $\lfloor \cdot \rfloor$  denotes the integer part). Recall that if  $n+1$  is odd, this representation is chosen so that the complex volume form acts as the identity on  $\Sigma_{n+1}$ .

Locally, by definition of  $\Sigma N$ , if  $U$  is an open subset of  $N$  and  $\psi \in \Gamma_U(\Sigma N)$  a local section of the spinor bundle, we can write

$$\psi = [\tilde{s}, \sigma]$$

where  $\sigma : U \rightarrow \Sigma_{n+1}$  and  $\tilde{s} : U \rightarrow \text{Spin} N$  are smooth maps, and  $[\tilde{s}, \sigma]$  is the equivalence class with respect to the relation

$$[\tilde{s}, \sigma] \sim [\tilde{s}g, \rho_{n+1}(g^{-1})\sigma], \quad \forall g \in \text{Spin}(n+1).$$

Moreover, we can always assume that  $\pi(\tilde{s})$  is a local section of  $\text{SON}$  with  $\nu$  for last basis vector. Then we have

$$\psi|_M = [(\tilde{s}|_{U \cap M}, \sigma|_{U \cap M})]$$

where the equivalence class is reduced to elements of  $\text{Spin}(n)$ .

It follows that one can realize the restriction to  $M$  of the spinor bundle  $\Sigma N$  as

$$S := \Sigma N|_M = \text{Spin} M \times_{\rho_{n+1} \circ \alpha} \Sigma_{n+1}.$$

**Remark.** The inclusion of  $\text{Spin}(n)$  in  $\text{Spin}(n+1)$  given by  $\alpha$  is the trivial one. But, this notation emphasizes that Clifford multiplication of a spinor field  $\phi \in \Gamma(S)$  by a vector  $X$  tangent to  $M$  is given by

$$(X, \phi) \mapsto X \cdot \nu \cdot \phi. \quad (4)$$

This fact is crucial for the following identification (see [1], [2]).

**2.2. Identification of  $S$  with  $\Sigma M$ .** We now compare  $S$  with the intrinsic spinor bundle of  $M$ ,

$$\Sigma M = \text{Spin} M \times_{\rho_n} \Sigma_n.$$

For this, we have to examine the cases where  $n$  is even or odd. First assume that  $n = 2m$  is even. From (3) and

$$\mathbb{C}l_{2m} \cong \mathbb{C}(2^m), \quad (5)$$

it follows that the representation of  $\mathbb{C}l_{2m}$  given by  $\rho_{2m+1} \circ \alpha$  is simply the restriction of  $\rho_{2m+1}$  to  $\mathbb{C}l_{2m+1}^0$ . But this representation is irreducible (see [9]).

The representation  $\rho_{2m+1} \circ \alpha$  is then an irreducible representation of  $\mathbb{C}l_{2m}$  of dimension  $\dim \Sigma_{2m+1} = 2^{\lceil \frac{2m+1}{2} \rceil} = 2^m$ , as  $\rho_{2m}$ . Now, (5) implies that such a representation is unique, up to an isomorphism. So  $\rho_{2m} \cong \rho_{2m+1} \circ \alpha$  and we can conclude that

$$S \cong \Sigma M. \quad (6)$$

Let  $\omega_{2m} = i^m e_1 \cdots e_{2m}$  be the complex volume form in even dimension. An easy calculation shows that  $\alpha(\omega) = \omega$ . The decomposition of  $\Sigma M$  into positive and negative parts is preserved under the isomorphism (6) and we have

$$S = S^+ \oplus S^-$$

where

$$S^\pm = \{\psi \in S \mid i\nu \cdot \psi = \pm \psi\} \cong \Sigma M^\pm.$$

Indeed, because we choose  $\rho_{2m+1}$  as the irreducible representation of  $\mathbb{C}l_{2m+1}$  for which the complex volume form  $\omega_{2m+1} = i^{m+1} e_1 \cdots e_{2m} \cdot \nu$  acts as the identity on  $\Sigma_{2m+1}$ , one has, for  $\psi \in S$ :

$$i\nu \cdot \psi = i\nu \cdot \omega_{2m+1} \cdot \psi = i^m i^2 \nu \cdot e_1 \cdots e_{2m} \cdot \nu \cdot \psi = \omega_{2m} \cdot \psi.$$

Assume now that  $n = 2m + 1$  is odd. Recall the following isomorphism:

$$\mathbb{C}l_{2m+1} = \mathbb{C}(2^m) \oplus \mathbb{C}(2^m). \quad (7)$$

As mentioned above,  $\rho_{2m+1}$  corresponds to the irreducible representation of  $\mathbb{C}l_{2m+1}$  for which the action of the complex volume form  $\omega_{2m+1}$  is the identity. Because  $n + 1 = 2m + 2$  is even,  $\Sigma N$  decomposes into positive and negative parts,

$$\Sigma N^\pm = \text{Spin} N \times_{\rho_{2m+2}^\pm} \Sigma_{2m+2}^\pm.$$

If  $e_k$  is a basis vector tangent to  $M$ , then

$$\begin{aligned} \alpha(e_k) \cdot \omega_{2m+2} &= i^{m+1} e_k \cdot \nu \cdot e_1 \cdots e_{2m+1} \cdot \nu \\ &= i^{m+1} (-1)^{2m+2} (-1)^{2m+2} e_1 \cdots e_{2m+1} \cdot \nu \cdot e_k \cdot \nu \\ &= \omega_{2m+2} \alpha(e_k). \end{aligned}$$

So  $\rho_{2m+2} \circ \alpha$  preserves the decomposition of  $\Sigma N$ , and

$$S = S^+ \oplus S^-$$

with

$$S^\pm = \text{Spin}M \times_{\rho_{2m+2}^\pm \circ \alpha} \Sigma_{2m+2}^\pm,$$

and where  $\omega_{2m+2}$  acts as  $\pm \text{Id}$  on  $S^\pm$ .

Moreover,

$$\alpha(\omega_{2m+1}) = i^{m+1}(e_1 \cdot \nu) \cdot \dots \cdot (e_{2m+1} \cdot \nu) = i^{m+1}e_1 \cdot \dots \cdot e_{2m+1} \cdot \nu = \omega_{2m+2},$$

and then  $\rho_{2m+1}$  and  $\rho_{2m+2}^+ \circ \alpha$  are both irreducible representations of  $\mathbb{C}l_{2m+1}$  of the same dimension, such that  $\rho_{2m+1}(\omega_{2m+1})$  and  $\rho_{2m+2}^+ \circ \alpha(\omega_{2m+1})$  are respectively the identity on  $\Sigma_{2m+1}$  and  $\Sigma_{2m+2}^+$ . Because such a representation is unique up to an isomorphism, we deduce that  $\rho_{2m+1} \cong \rho_{2m+2}^+ \circ \alpha$  and

$$S^+ \cong \Sigma M. \quad (8)$$

Thus we have shown the following

**Proposition 2.1.** *If  $n$  is even (resp. odd), there exists an identification of the hypersurface spinor bundle  $S$  (resp.  $S^+$ ) with the spinor bundle  $\Sigma M$  which sends every spinor  $\varphi \in S$  (resp.  $S^+$ ) to the spinor denoted by  $\varphi^* \in \Sigma M$ . Moreover, with respect to this identification, Clifford multiplication by a vector field  $X$ , tangent to  $M$ , is given by*

$$X \cdot \varphi^* = (X \cdot \nu \cdot \varphi)^*.$$

**2.3. The Spinorial Gauss Formula and the Hypersurface Dirac Operator.** Let  $\tilde{\nabla}$  be the Levi-Civita connection of  $(N^{n+1}, \tilde{g})$ , and  $\nabla$  that of  $(M^n, g)$ . Let  $(e_1, \dots, e_n, e_{n+1} = \nu)$  be a local orthonormal basis for  $TM$ , then the Gauss formula says that for  $1 \leq i, j \leq n$ ,

$$\tilde{\nabla}_i e_j = \nabla_i e_j + h_{ij} \nu, \quad (9)$$

where  $h_{ij}$  are the coefficients of the second fundamental form of the hypersurface  $M$ . We are going to relate the associated connections on the corresponding spinor bundles. For this, consider  $\phi \in \Gamma(\Sigma N)$  and  $\varphi = \phi|_M \in \Gamma(S)$  its restriction to  $M$ . Recall now that locally, for  $X \in \Gamma(TM)$ ,

$$\tilde{\nabla}_X \phi = X(\phi) + \frac{1}{2} \sum_{1 \leq i < j \leq n+1} \tilde{g}(\tilde{\nabla}_X e_i, e_j) e_i \cdot e_j \cdot \phi \quad (10)$$

and

$$\begin{aligned}
\nabla_X \varphi &= X(\varphi) + \frac{1}{2} \sum_{1 \leq i < j \leq n} g(\nabla_X e_i, e_j) e_i \cdot \nu \cdot e_j \cdot \nu \cdot \varphi \\
&= X(\varphi) + \frac{1}{2} \sum_{1 \leq i < j \leq n} g(\nabla_X e_i, e_j) e_i \cdot e_j \cdot \varphi.
\end{aligned}$$

Therefore, by restricting both sides of equation (10) to  $M$ , and using the fact that  $X(\phi)|_M = X(\phi|_M)$  for  $X$  tangent to  $M$ , the Gauss formula (9) yields, for  $1 \leq k \leq n$ ,

$$\begin{aligned}
(\tilde{\nabla}_k \phi)|_M &= e_k(\varphi) + \frac{1}{2} \sum_{1 \leq i < j \leq n} \tilde{g}(\nabla_k e_i + h_{ki} \nu, e_j) e_i \cdot e_j \cdot \varphi \\
&\quad + \frac{1}{2} \sum_{1 \leq i \leq n} \tilde{g}(\nabla_k e_i + h_{ki} \nu, \nu) e_i \cdot \nu \cdot \varphi \\
&= e_k(\varphi) + \frac{1}{2} \sum_{1 \leq i < j \leq n} g(\nabla_k e_i, e_j) e_i \cdot e_j \cdot \varphi \\
&\quad + \frac{1}{2} \sum_{1 \leq i \leq n} h_{ki} e_i \cdot \nu \cdot \varphi \\
&= \nabla_k \varphi + \frac{1}{2} \sum_{1 \leq i \leq n} h_{ki} e_i \cdot \nu \cdot \varphi
\end{aligned}$$

Once again, from equation (10), it makes sense to write  $(\tilde{\nabla}_X \phi)|_M = \tilde{\nabla}_X \varphi$  when  $X$  is tangent to  $M$ , and hence we proved the spinorial Gauss formula:

$$\forall \varphi \in \Gamma(S), \forall X \in \Gamma(TM), \quad \tilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} h(X) \cdot \nu \cdot \varphi. \quad (11)$$

(Here  $h$  is seen as an endomorphism of the tangent bundle.)

It is known (see [9]) that there exists a positive definite Hermitian metric  $\langle \cdot, \cdot \rangle$  on  $\Sigma N$  such that, if  $\tau$  is a  $k$ -form on  $N$ ,

$$\langle \tau \cdot \phi, \psi \rangle = (-1)^{\frac{k(k+1)}{2}} \langle \phi, \tau \cdot \psi \rangle, \quad \forall \phi, \psi \in \Gamma(\Sigma N). \quad (12)$$

If we denote  $(\cdot, \cdot)$  its real part, we have

$$(\tilde{X} \cdot \phi, \tilde{Y} \cdot \phi) = \tilde{g}(\tilde{X}, \tilde{Y})(\phi, \phi) \quad , \quad (\tilde{X} \cdot \phi, \phi) = 0 \quad , \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(TN). \quad (13)$$

We simply restrict  $(.,.)$  to  $M$  to get a globally defined metric on  $S$ . Now, because  $\tilde{\nabla}$  is compatible with  $(.,.)$ , i.e.

$$X(\varphi, \psi) = (\tilde{\nabla}_X \varphi, \psi) + (\varphi, \tilde{\nabla}_X \psi) \quad , \quad \forall \varphi, \psi \in \Gamma(S), \forall X \in \Gamma(TM)$$

formula (11) easily implies that  $\nabla$  is also compatible with the metric. We remark that equation (11) implies that with respect to the identification of proposition 2.1, we have

$$(\nabla \phi)^* = \nabla \phi^* \tag{14}$$

This leads to the metric  $(.,.)_{\Sigma M}$  on the intrinsic spinor bundle, with the same properties as  $(.,.)$ , and hence the two bundles are isometric.

Because Clifford multiplication of a spinor by a vector tangent to  $M$  is given by (4), if  $n$  is odd,  $S^+$  is stable by  $\nabla$  and by Clifford multiplication. So the classical Dirac operator is simply defined on  $S$  for  $n$  even (resp.  $S^+$  for  $n$  odd) by

$$D = \sum_{i=1}^n e_i \cdot \nu \cdot \nabla_i.$$

Now we define the hypersurface Dirac operator on  $\Gamma(S)$  as

$$D_H = \sum_{i=1}^n e_i \cdot \nu \cdot \tilde{\nabla}_i.$$

This definition is motivated by the following fact. Let

$$\tilde{D} = \sum_{i=1}^n e_i \cdot \tilde{\nabla}_i$$

be the hypersurface Dirac operator defined by Witten (see [11], [10]) to prove the positive energy conjecture in general relativity. Then  $\tilde{D}$  is not formally self-adjoint with respect to the metric  $(.,.)$ . Indeed, it is proved in [8] that

$$D_H^2 = \tilde{D}^* \tilde{D},$$

where  $\tilde{D}^*$  is the formal adjoint of  $\tilde{D}$  w.r.t.  $(.,.)$ .



From formula (11), we see that for  $n$  even (resp. odd), we have the following relations on  $\Gamma(S)$  (resp.  $\Gamma(S^+)$ ) :

$$\begin{aligned}
D_H &= \sum_i e_i \cdot \nu \cdot \nabla_i + \sum_i e_i \cdot \nu \cdot \frac{h(e_i)}{2} \cdot \nu \cdot \\
&= D + \sum_{i,j} \frac{h_{ij}}{2} e_i \cdot e_j \cdot \\
&= D + \sum_{i,j} \frac{h_{ij}}{4} (e_i \cdot e_j + e_j \cdot e_i) \cdot \\
&= D - \sum_{i,j} \frac{h_{ij}}{2} \delta_{ij}
\end{aligned}$$

and hence, if  $H = \sum_i h_{ii}$  is the mean curvature of the hypersurface, we have

$$D_H = D - \frac{H}{2} \quad (15)$$

In the following, we will not distinguish the cases where  $n$  is even or odd. In fact, if  $n$  is odd,  $D_H$  preserves the decomposition of  $S$  into positive and negative spinors, as well as Clifford multiplication (recall (4)),  $\tilde{\nabla}$  and  $\nabla$ . Indeed, if  $\phi \in \Gamma(S)$  is an eigenspinor of  $D_H$  with eigenvalue  $\lambda$ , it is the same for  $\phi^+$ , its positive part. So we only consider positive spinors. The notation becomes easier with this convention.

Now, it is easy to see from equation (15) that  $D_H$  is formally self-adjoint with respect to the metric  $(\cdot, \cdot)$  (see [8]). Finally, recall the well-known Schrödinger-Lichnerowicz formula on  $\Gamma(\Sigma M)$  which by the previous identification is also true on  $\Gamma(S)$ :

$$D^2 = \nabla^* \nabla + \frac{R}{4}, \quad (16)$$

$R$  being the scalar curvature of  $M$  and  $\nabla^*$  the formal adjoint of  $\nabla$  with respect to the metric  $(\cdot, \cdot)$ .

### 3. PROOF OF THE THEOREM 1.1

Now we give an estimate for the eigenvalues of  $D_H$  in terms of the energy-momentum tensor (see [7]). For any spinor field  $\varphi \in \Gamma(S)$ , we define the

associated energy-momentum 2-tensor  $Q^\varphi$  on the complement of its zero set, by

$$Q_{ij}^\varphi = \frac{1}{2}(e_i \cdot \nu \cdot \nabla_j \varphi + e_j \cdot \nu \cdot \nabla_i \varphi, \varphi / |\varphi|^2). \quad (17)$$

*Remark 3.1.* This definition corresponds to the one given in [7] if we note that with respect to the identification of  $S$  with  $\Sigma M$  of proposition 2.1,

$$Q_{ij}^\varphi = \frac{1}{2}(e_i \cdot \nabla_j \varphi^* + e_j \cdot \nabla_i \varphi^*, \varphi^* / |\varphi^*|^2)_{\Sigma M}.$$

If  $\varphi$  is an eigenspinor for  $D_H$ ,  $Q^\varphi$  is well defined in the sense of distribution. For any real functions  $p$  and  $q$ , consider the modified covariant derivative defined on  $S$  by

$$\nabla_i^Q = \nabla_i + \left(p \frac{H}{2} + q \lambda\right) e_i \cdot \nu \cdot + \sum_j Q_{ij}^\varphi e_j \cdot \nu \cdot. \quad (18)$$

*Remark 3.2.* This connection is well defined on  $S^+$  when  $n$  is odd.

Using (13), we have

$$\begin{aligned} |\nabla^Q \varphi|^2 &= |\nabla \varphi|^2 + n \left(p \frac{H}{2} + q \lambda\right)^2 |\varphi|^2 \\ &\quad + \sum_{i,j,k} Q_{ij}^\varphi Q_{ik}^\varphi (e_j \cdot \nu \cdot \varphi, e_k \cdot \nu \cdot \varphi) \\ &\quad + 2 \left(p \frac{H}{2} + q \lambda\right) \sum_i (\nabla_i \varphi, e_i \cdot \nu \cdot \varphi) \\ &\quad + 2 \sum_{i,j} Q_{ij}^\varphi (\nabla_i \varphi, e_j \cdot \nu \cdot \varphi) \\ &\quad + 2 \left(p \frac{H}{2} + q \lambda\right) \sum_{i,j} Q_{ij}^\varphi (e_i \cdot \nu \cdot \varphi, e_j \cdot \nu \cdot \varphi) \\ &= |\nabla \varphi|^2 + n \left(p \frac{H}{2} + q \lambda\right)^2 |\varphi|^2 + |Q^\varphi|^2 |\varphi|^2 \\ &\quad - 2 \left(p \frac{H}{2} + q \lambda\right) (D\varphi, \varphi) - 2 |Q^\varphi|^2 |\varphi|^2 \\ &\quad + 2 \left(p \frac{H}{2} + q \lambda\right) \text{Tr}(Q^\varphi) |\varphi|^2 \end{aligned}$$

but

$$\text{Tr}(Q^\varphi) |\varphi|^2 = (D\varphi, \varphi)$$

hence

$$|\nabla^Q \varphi|^2 = |\nabla \varphi|^2 + n \left( p \frac{H}{2} + q \lambda \right)^2 |\varphi|^2 - |Q^\varphi|^2 |\varphi|^2. \quad (19)$$

Now, since  $D_H = D - \frac{H}{2}$ , the Schrödinger-Lichnerowicz formula (16) on  $\Gamma(S)$  gives

$$\begin{aligned} \int_M |\nabla \varphi|^2 v_g &= \int_M \left( |D\varphi|^2 - \frac{R}{4} |\varphi|^2 \right) v_g \\ &= \int_M \left( \left( \lambda + \frac{H}{2} \right)^2 - \frac{R}{4} \right) |\varphi|^2 v_g \end{aligned} \quad (20)$$

Therefore (19) and (20) imply

$$\begin{aligned} \int_M |\nabla^Q \varphi|^2 v_g &= \int_M \left( (1 + nq^2) \lambda^2 - \frac{R}{4} - |Q^\varphi|^2 \right) |\varphi|^2 v_g \\ &\quad + \int_M \left( (1 + np^2) \frac{H^2}{4} + (1 + npq) H \lambda \right) |\varphi|^2 v_g. \end{aligned} \quad (21)$$

Now, assume that  $q$  has no zeros so that we can choose  $p = -\frac{1}{nq}$ . Then (21) becomes

$$\int_M |\nabla^Q \varphi|^2 v_g = \int_M (1 + nq^2) \left[ \lambda^2 - \frac{1}{4} \left( \frac{R + 4|Q^\varphi|^2}{(1 + nq^2)} - \frac{H^2}{nq^2} \right) \right] |\varphi|^2 v_g \quad (22)$$

If  $R + 4|Q^\varphi|^2 > H^2 > 0$ , we can take

$$nq^2 = \frac{|H|}{\sqrt{R + 4|Q^\varphi|^2} - |H|} \quad (23)$$

Then equation (22) becomes

$$\int_M |\nabla^Q \varphi|^2 v_g = \int_M (1 + nq^2) \left[ \lambda^2 - \frac{1}{4} \left( \sqrt{R + 4|Q^\varphi|^2} - |H| \right)^2 \right] |\varphi|^2 v_g \quad (24)$$

Because the left hand side of this equation is positive and  $\lambda$  is a constant, we get

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{R + 4|Q^\varphi|^2} - |H| \right)^2. \quad (25)$$

*Remark 3.3.* If  $M$  is minimal, i.e.  $H = 0$ , we can choose  $q \equiv 0$  in (18) so that (25) specializes to the inequality of Theorem A in [7].

*Remark 3.4.* Note that our definition of the energy-momentum tensor  $Q^\varphi$  coincides with that in [7]. The definition used in [12] and [8] gives a factor  $\frac{n}{n-1}$  in front of  $R + 4|Q^\varphi|^2$  in inequality (25) but in this case,  $Q^\varphi$  has no canonical intrinsic meaning.

#### 4. LIMITING CASES

First recall the inequality proved by X. Zhang :

**Theorem 4.1** ([12]). *Let  $M^n \subset N^{n+1}$  be a compact hypersurface of a riemannian spin manifold  $(N, \tilde{g})$ . Assume that  $n \geq 2$  and  $nR > (n-1)H^2 > 0$ . Then if  $\lambda$  is any eigenvalue of the hypersurface Dirac operator  $D_H$ , one has*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{\frac{n}{n-1}R} - |H| \right)^2. \quad (26)$$

As in the proof of Theorem 1.1, the proof of Theorem 4.1 is based on the use of the modified connection

$$\nabla_i^\lambda = \nabla_i + \left( p \frac{H}{2} + q \lambda \right) e_i \cdot \nu. \quad (27)$$

Here,  $p$  and  $q$  are related by

$$p = \frac{1-q}{1-nq} \quad (28)$$

and

$$q = \frac{1}{n} \left( 1 - \sqrt{\frac{(n-1)|H|}{\sqrt{\frac{n}{n-1}R} - |H|}} \right) \quad (29)$$

or, in other terms,

$$(1-nq)^2 = \frac{(n-1)|H|}{\sqrt{\frac{n}{n-1}R} - |H|}. \quad (30)$$

Equality holds in (26) for an eigenspinor  $\varphi$  of  $D_H$  with eigenvalue  $\lambda$  if and only if  $\sqrt{\frac{n}{n-1}R} - |H|$  is constant and  $\nabla^\lambda \varphi \equiv 0$ . But, with respect to the identification of proposition 2.1, and by (14),  $\nabla^\lambda \varphi \equiv 0$  is equivalent to

$$\forall i = 1, \dots, n, \quad \nabla_i \varphi^* = - \left( p \frac{H}{2} + q \lambda \right) e_i \cdot \varphi^* \quad (31)$$

It is known (see [6]) that if such a section exists on  $\Sigma M$ , then  $p \frac{H}{2} + q \lambda$  has to be constant (say  $\frac{\lambda_1}{n}$  for instance) and that in this case  $M$  is Einstein and

$R = 4\frac{n-1}{n}\lambda_1^2$ . So  $\varphi$  is a Killing spinor and we are in the limiting case of Friedrich's inequality [3]. Moreover, since  $\sqrt{\frac{n}{n-1}R} - |H|$  is constant,  $H$  has to be constant.

Therefore, since  $D\varphi = \lambda_1\varphi$  and  $\lambda_1 = \frac{\text{sign}(\lambda_1)}{2}\sqrt{\frac{n}{n-1}R}$ , the following equation must be satisfied (recall that  $D_H = D - \frac{H}{2}$ )

$$\lambda = \frac{\text{sign}(\lambda_1)}{2}\sqrt{\frac{n}{n-1}R} - \frac{H}{2} = \frac{\text{sign}(\lambda_1)}{2}\sqrt{\frac{n}{n-1}R} - \text{sign}(H)\frac{|H|}{2} \quad (32)$$

But equality case gives

$$\lambda = \frac{\text{sign}(\lambda)}{2}\left(\sqrt{\frac{n}{n-1}R} - |H|\right), \quad (33)$$

So (32) and (33) imply that

$$\text{sign}(\lambda) = \text{sign}(\lambda_1) = \text{sign}(H) \quad (34)$$

On the other hand, an easy calculation leads to

$$\begin{aligned} p\frac{H}{2} + q\lambda &= \frac{\text{sign}(\lambda)}{2n}\sqrt{\frac{n}{n-1}R} \\ &\quad + \frac{(\text{sign}(H) - \text{sign}(\lambda))}{2n}\left(1 + \sqrt{(n-1)\left(\sqrt{\frac{n}{n-1}R} - |H|\right)}\right) \\ &= \frac{\text{sign}(\lambda_1)}{2n}\sqrt{\frac{n}{n-1}R} \end{aligned}$$

and we recover the already known fact that  $p\frac{H}{2} + q\lambda = \frac{\lambda_1}{n}$ .

Indeed, (34) can be trivially observed because in the equality case, both  $R$  and  $H$  are constant, so we can think of the spectrum of  $D_H$  as the shifting of the spectrum of  $D$  by  $-\frac{H}{2}$ . Then the condition  $nR > (n-1)H^2 > 0$  in Theorem 4.1 simply implies that the lowest eigenvalue of  $D_H$  (in the sense of its absolute value) must have the sign of  $H$ . In particular, when  $n$  is even, it shows how we lose the symmetry of the spectrum when passing from  $D$  to  $D_H$  (compare with the case where  $H = 0$ ).

Now we discuss the case of Theorem 1.1. The limiting case of inequality (1) holds for an eigenspinor  $\varphi$  of  $D_H$  with eigenvalue  $\lambda$  if and only if  $\nabla^Q\varphi \equiv 0$ . First note that this implies that  $|\varphi|$  is constant. Then, with respect to the

identification of proposition 2.1, and by (14),  $\nabla^Q \varphi \equiv 0$  is equivalent to

$$\nabla_i \varphi^* = -\left(p \frac{H}{2} + q \lambda\right) e_i \cdot \varphi^* - \sum_j Q_{ij}^\varphi e_j \cdot \varphi^* \quad (35)$$

Let  $f = p \frac{H}{2} + q \lambda$ , then equation (35) can be written as

$$\nabla_i \varphi^* = - \sum_j (Q_{ij}^\varphi + f \delta_{ij}) e_j \cdot \varphi^* \quad (36)$$

Now let  $T_{ij} = Q_{ij}^\varphi + f \delta_{ij}$ , taking Clifford multiplication by  $e_k$  on both sides of equation (36), yields

$$e_k \cdot \nabla_i \varphi^* = - \sum_j T_{ij} e_k \cdot e_j \cdot \varphi^*,$$

which gives

$$(e_k \cdot \nabla_i \varphi^*, \varphi^*)_{\Sigma M} = - \sum_j T_{ij} (e_k \cdot e_j \cdot \varphi^*, \varphi^*)_{\Sigma M},$$

and, because  $(e_k \cdot e_j \cdot \varphi^*, \varphi^*)_{\Sigma M} = 0$  unless  $j = k$  and  $T_{ij}$  is symmetric, we proved

$$\frac{1}{2} (e_i \cdot \nabla_k \varphi^* + e_k \cdot \nabla_i \varphi^*, \varphi^* / |\varphi^*|^2)_{\Sigma M} = T_{ik}.$$

Hence

$$T_{ik} = Q_{ik}^\varphi$$

and we can conclude that  $f = 0$ . Equation (35) reduces to

$$\nabla_i \varphi^* = - \sum_j Q_{ij}^\varphi e_j \cdot \varphi^*. \quad (37)$$

Such field equations have been studied, as well as their integrability conditions, by T. Friedrich and E. C. Kim in [5]. Note that they allow a nice formulation of the theory of immersed surfaces in the euclidean 3-space (see [3]). We will call an EM-spinor (for Energy-Momentum spinor) a non trivial spinor field satisfying (37). If it is an eigenspinor for the Dirac operator, which is equivalent to the fact that  $\text{tr } Q^\varphi$  is constant, it is called T-Killing spinor (see [4]). In fact, a T-Killing spinor is exactly a spinor field satisfying the limiting case in Hijazi's inequality [7].

Now we have (see [7] or Lemma 4.1(iii) of [5])

$$(\text{tr } Q^\varphi)^2 = \frac{R}{4} + |Q^\varphi|^2 \quad (38)$$

So (37) implies that

$$D\varphi = F\varphi$$

where  $F^2 = \frac{R}{4} + |Q^\varphi|^2$ . Whereas equality case in (1) gives  $\sqrt{R + 4|Q^\varphi|^2} - |H|$  is constant, we can't conclude here that  $\frac{R}{4} + |Q^\varphi|^2$  and  $H$  are constant as in the case of Zhang's inequality. Nevertheless, we have the following

**Corollary 4.2.** *If  $H$  is constant, then equality case in (1) holds if and only if  $\varphi$  is a  $T$ -Killing spinor.*

By hypothesis  $H$  has constant sign and we can conclude that  $\lambda$  has the same sign. Recall that  $p$  and  $q$  are related by

$$p = -\frac{1}{nq}$$

and

$$nq^2 = \frac{|H|}{\sqrt{R + 4|Q^\varphi|^2} - |H|}.$$

Indeed, an easy calculation gives

$$0 = f = \left(p \frac{H}{2} + q \lambda\right) = \frac{(\text{sign}(\lambda) - \text{sign}(H))}{2\sqrt{n}} \sqrt{|H|(\sqrt{R + 4|Q^\varphi|^2} - |H|)}. \quad (39)$$

Hence

$$\text{sign}(\lambda) = \text{sign}(H).$$

*Remark 4.3.* Equality case of (26) is included in that of (1): if we assume that  $\varphi$  is a Killing spinor, then necessarily  $Q_{ij}^\varphi = \frac{\lambda}{n} \delta_{ij}$  and so  $(\text{tr } Q^\varphi)^2 = \lambda_1^2 = \frac{1}{4} \frac{n}{n-1} R$ . Therefore equation (38) implies

$$4|Q^\varphi|^2 = \frac{n}{n-1} R - R$$

and we have

$$\lambda^2 = \left( \sqrt{\frac{n}{n-1} R} - |H| \right)^2.$$

*Remark 4.4.* The previous remark shows that Theorem 1.1 improves Theorem 4.1. In particular, it does not require  $R$  to be positive, and the limiting case does not imply that  $H$  has to be constant.

## 5. PROOF OF THE THEOREM 1.2

Consider a conformal change of the metric  $\bar{g} = e^{2u}\tilde{g}$  for any real function  $u$  on  $N$ . For simplicity, let  $\bar{N} = (N, \bar{g})$ . The natural isometry between  $\text{SON}$  and  $\text{SO}\bar{N}$  induced by this conformal change of the metric lifts to an isometry between the  $\text{Spin}(n+1)$ -principal bundles  $\text{Spin}N$  and  $\text{Spin}\bar{N}$ , and hence between the two corresponding hypersurface spinor bundles  $S$  and  $\bar{S}$ . If  $\varphi \in \Gamma(S)$ , denote by  $\bar{\varphi} \in \Gamma(\bar{S})$  its image by this isometry. Let  $(\cdot, \cdot)_{\bar{g}}$  be the metric on  $\bar{S}$  naturally defined as described in section 2. Then for  $\varphi, \psi$  two sections of  $S$ , we have

$$(\varphi, \psi) = (\bar{\varphi}, \bar{\psi})_{\bar{g}} \quad \text{and} \quad \overline{X \cdot \psi} = \overline{X} \cdot \bar{\psi}$$

We will also denote by  $\bar{g} = e^{2u}|_M g$  the restriction of  $\bar{g}$  to  $M$ . By conformal covariance of the Dirac operator, we have, for  $\varphi \in \Gamma(S)$ , (see [8])

$$\bar{D} \left( e^{-\frac{(n-1)}{2}u} \bar{\varphi} \right) = e^{-\frac{(n+1)}{2}u} \bar{D}\bar{\varphi}, \quad (40)$$

where  $\bar{D}$  stands for the Dirac operator w.r.t. to  $\bar{g}$ . On the other hand

$$\bar{H} = e^{-u} \left( H + n \, du(\nu) \right). \quad (41)$$

Therefore, if  $\bar{D}_{\bar{H}}$  stands for the hypersurface Dirac operator w.r.t. to  $\bar{g}$ , equations (40) and (41) imply that,

$$\bar{D}_{\bar{H}} \left( e^{-\frac{(n-1)}{2}u} \bar{\varphi} \right) = e^{-\frac{(n+1)}{2}u} \left( \overline{D_H \varphi} - \frac{n}{2} du(\nu) \bar{\varphi} \right). \quad (42)$$

*Remark 5.1.* We see that if  $du(\nu)|_M = 0$ ,  $D_H$  is a conformal invariant operator. In this case, technics used in ([6]) can be applied for the eigenvalues of  $D_H$ . Indeed, such a conformal change of metric can be viewed as a intrinsic conformal change of the metric on  $M$ , when we omit the ambient space  $N$  (See section 7).

From now on, we will only consider conformal changes of the metric  $\bar{g} = e^{2u}\tilde{g}$  with  $du(\nu) = 0$  on  $M$ . They will be called regular conformal changes of metric as in [8].

For  $\varphi \in \Gamma(S)$  an eigenspinor of  $D_H$  with eigenvalue  $\lambda$ , let  $\bar{\psi} := e^{-\frac{n-1}{2}u} \bar{\varphi}$ . Then (42) gives

$$\bar{D}_{\bar{H}} \bar{\psi} = \lambda e^{-u} \bar{\psi}. \quad (43)$$



Recall that

$$\overline{\nabla}_i \overline{\varphi} = \overline{\nabla}_i \varphi - \frac{1}{2} \overline{e_i \cdot du \cdot \varphi} - \frac{1}{2} e_i(u) \overline{\varphi}, \quad (44)$$

and  $\overline{e}_i = e^{-u} e_i$ . Now, as in [6], it is straightforward to get

$$\begin{aligned} \overline{Q}_{ij}^{\overline{\psi}} &= \frac{1}{2} \left( \overline{e}_i \cdot \overline{\nu} \cdot \overline{\nabla}_{\overline{e}_j} \overline{\psi} + \overline{e}_j \cdot \overline{\nu} \cdot \overline{\nabla}_{\overline{e}_i} \overline{\psi}, \overline{\psi} / |\overline{\psi}|_{\overline{g}}^2 \right)_{\overline{g}} \\ &= \frac{1}{2} e^{-u} \left( \overline{e}_i \cdot \overline{\nu} \cdot \overline{\nabla}_{e_j} \overline{\varphi} + \overline{e}_j \cdot \overline{\nu} \cdot \overline{\nabla}_{e_i} \overline{\varphi}, \overline{\varphi} / |\overline{\varphi}|_{\overline{g}}^2 \right)_{\overline{g}} \\ &= \frac{1}{2} e^{-u} \left( e_i \cdot \nu \cdot \nabla_{e_j} \varphi + e_j \cdot \nu \cdot \nabla_{e_i} \varphi, \varphi / |\varphi|^2 \right) \\ &= e^{-u} Q_{ij}^{\varphi}. \end{aligned} \quad (45)$$

Hence,

$$|\overline{Q}^{\overline{\psi}}|^2 = e^{-2u} |Q^{\varphi}|^2 \quad (46)$$

Equation (22), which is also true on  $N$ , applied to  $\overline{\psi}$  yields

$$\int_M |\overline{\nabla}^Q \overline{\psi}|^2 v_{\overline{g}} = \int_M (1 + nq^2) \left[ \lambda^2 e^{-2u} - \frac{1}{4} \left( \frac{\overline{R} + 4|\overline{Q}^{\overline{\psi}}|^2}{(1 + nq^2)} - \frac{\overline{H}^2}{nq^2} \right) \right] |\overline{\psi}|^2 v_{\overline{g}} \quad (47)$$

which, because of (41) and (46) gives

$$\int_M |\overline{\nabla}^Q \overline{\psi}|^2 v_{\overline{g}} = \int_M (1 + nq^2) \left[ \lambda^2 - \frac{1}{4} \left( \frac{\overline{R} e^{2u} + 4|Q^{\varphi}|^2}{(1 + nq^2)} - \frac{H^2}{nq^2} \right) \right] e^{-2u} |\overline{\psi}|^2 v_{\overline{g}}. \quad (48)$$

Taking

$$nq^2 = \frac{|H|}{\sqrt{\overline{R} e^{2u} + 4|Q^{\varphi}|^2} - |H|}$$

completes the proof of Theorem 1.2.

## 6. GENERAL LIMITING CASES

We now discuss the limiting case in inequality (2). Equality holds if and only if  $\overline{\nabla}_i^Q \overline{\psi} = 0$  for  $1 \leq i \leq n$ , which can be written as

$$0 = \overline{\nabla}_i \overline{\psi} + \left( p \frac{\overline{H}}{2} + q e^{-u} \lambda \right) \overline{e}_i \cdot \overline{\nu} \cdot \overline{\psi} + \sum_j \overline{Q}_{ij}^{\overline{\psi}} \overline{e}_j \cdot \overline{\nu} \cdot \overline{\psi}.$$

Since  $\bar{\psi} := e^{-\frac{n-1}{2}u} \bar{\varphi}$ , (44) and (46) yield

$$\begin{aligned} 0 &= e^{-\frac{n-1}{2}u} e^{-u} \left[ \overline{\nabla_i \varphi} - \frac{1}{2} \overline{e_i \cdot du \cdot \varphi} - \frac{n}{2} e_i(u) \bar{\varphi} \right. \\ &\quad \left. + \left( p \frac{H}{2} + q \lambda \right) \overline{e_i \cdot \nu \cdot \varphi} + \sum_j Q_{ij}^\varphi \overline{e_j \cdot \nu \cdot \varphi} \right]. \end{aligned} \quad (49)$$

With respect to the identification of proposition 2.1, and by (14), this last statement is equivalent to

$$\nabla_i \varphi^* = \frac{1}{2} e_i \cdot du \cdot \varphi^* + \frac{n}{2} du(e_i) \varphi^* - f e_i \cdot \varphi^* - \sum_j Q_{ij}^\varphi e_j \cdot \varphi^*. \quad (50)$$

where  $f := p \frac{H}{2} + q \lambda$ . As in Section 4, let  $T_{ij} = Q_{ij}^\varphi + f \delta_{ij}$ . It is then straightforward to prove that  $T_{ij} = Q_{ij}^\varphi$  and so  $f = 0$ .

Taking the scalar product of (50) with  $\varphi^*$ , it follows

$$\begin{aligned} \frac{1}{2} e_i(|\varphi|^2) &= (\nabla_i \varphi^*, \varphi^*)_{\Sigma M} \\ &= \frac{1}{2} (e_i \cdot du \cdot \varphi^*, \varphi^*)_{\Sigma M} + \frac{n}{2} du(e_i) |\varphi|^2 \\ &= \frac{(n-1)}{2} du(e_i) |\varphi|^2 \end{aligned}$$

Therefore,

$$du = \frac{d|\varphi|^2}{(n-1)|\varphi|^2}. \quad (51)$$

So we proved that equality holds in (2) if and only if the eigenspinor  $\varphi$  satisfies

$$\nabla_i \varphi^* = \frac{1}{2} e_i \cdot du \cdot \varphi^* + \frac{n}{2} du(e_i) \varphi^* - \sum_j Q_{ij}^\varphi e_j \cdot \varphi^*$$

with  $u$  satisfying (51). Such field equations have already been studied, as well as their integrability conditions, by T. Friedrich and E.C. Kim [5]. We will call them WEM-spinors (for Weak Energy-Momentum spinors). If, they satisfy the Einstein-Dirac equation, they are called WK-spinors (for Weak Killing spinors). These are exactly the limiting case of Hijazi's equality involving conformal change of the metric and the energy-momentum tensor [7], in which case, they are also eigenspinors for the classical Dirac operator.

In our situation, there are not eigenspinors for  $D$ . As a consequence, even if in the limiting case  $\sqrt{\overline{R}e^{2u} + 4|Q^\varphi|^2} - |H|$  has to be constant, we can't conclude that both  $\sqrt{\overline{R}e^{2u} + 4|Q^\varphi|^2}$  and  $H$  are constant.

Nevertheless, as in the previous section, a simple calculation leads to

$$0 = f = e^u \frac{(\text{sign}(\lambda) - \text{sign}(H))}{2\sqrt{n}} \sqrt{|H|(\sqrt{\overline{R}e^{2u} + 4|Q^\varphi|^2} - |H|)}.$$

Hence

$$\text{sign}(\lambda) = \text{sign}(H).$$

Now recall the inequality proved by O. Hijazi and X. Zhang :

**Theorem 6.1** ([8]). *Let  $M^n \subset N^{n+1}$  be a compact hypersurface of a Riemannian spin manifold  $(N, \tilde{g})$ . Assume that  $n \geq 2$  and  $n\overline{R}e^{2u} > (n-1)H^2 > 0$  for some regular conformal change of the metric  $\overline{g} = e^{2u}\tilde{g}$ . Then if  $\lambda$  is any eigenvalue of the hypersurface Dirac operator  $D_H$ , one has*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{\frac{n}{n-1} \overline{R}e^{2u}} - |H| \right)^2. \quad (52)$$

As in the proof of Theorem 1.2, Theorem 6.1 is obtained by using the modified connection defined by (27), on the manifold  $(N, \overline{g} = e^{2u}\tilde{g})$ .

As in the beginning of this section, it is then easy to see that equality holds in (52) if and only if

$$0 = e^{-\frac{n-1}{2}u} e^{-u} \left[ \overline{\nabla_i \varphi} - \frac{1}{2} \overline{e_i \cdot du \cdot \varphi} - \frac{n}{2} e_i(u) \overline{\varphi} + \left( p \frac{H}{2} + q \lambda \right) \overline{e_i \cdot \nu \cdot \varphi} \right]. \quad (53)$$

With respect to the identification of proposition 2.1, and by (14), this last statement is equivalent to

$$\nabla_i \varphi^* = \frac{1}{2} e_i \cdot du \cdot \varphi^* + \frac{n}{2} du(e_i) \varphi^* - f e_i \cdot \varphi^*. \quad (54)$$

where  $f := p \frac{H}{2} + q \lambda$ . As in section 4, let  $T_{ij} = f \delta_{ij}$ . Then it is straightforward to prove that  $T_{ij} = Q_{ij}^\varphi$  and that spinors fields satisfying the equality case in Theorem 6.1 are particular WEM-spinors. Now, by (38) and by (45), we see that necessarily

$$f = \pm \frac{1}{n} \sqrt{\frac{n}{n-1} \overline{R}e^{2u}}. \quad (55)$$

Hence, solutions of (54) correspond exactly to sections verifying the limiting case of inequality (5.1) in [6].

Recall that here,  $p$  and  $q$  are given by

$$p = \frac{1 - q}{1 - nq} \quad (56)$$

and

$$(1 - nq)^2 = \frac{(n - 1)|H|}{\sqrt{\frac{n}{n-1} \overline{R} e^{2u} - |H|}}. \quad (57)$$

Therefore we can recover that  $\text{sign}(\lambda) = \text{sign}(H)$  as made previously by computing explicitly  $p \frac{H}{2} + q\lambda$ . In fact, as in Remark 4.3, the consequence of equation (38) is that Theorem 1.2 improves Theorem 6.1.

## 7. CONCLUDING REMARK

We conclude this paper by observing that all computations previously made could be done in an intrinsic way, considering a modified Dirac operator  $D_f = D - \frac{f}{2}$ , and connections on  $\Sigma M$ :

$$\nabla_i^\lambda = \nabla_i + \left( p \frac{f}{2} + q \lambda \right) e_i \cdot$$

and

$$\nabla_i^Q = \nabla_i + \left( p \frac{f}{2} + q \lambda \right) e_i \cdot + Q_{ij}^\varphi e_j \cdot,$$

with an appropriate choice of  $p$  and  $q$  (simply replace  $H$  by  $f$ ), in (28) and (30).

The identification of the spinor bundles of section 2 allows to assert that computations will lead to the same results, but in a more general way, and so we proved

**Proposition 7.1.** *Let  $(M^n, g)$  be a compact Riemannian spin manifold. Assume that  $n \geq 2$  and  $nR > (n - 1)f^2 > 0$ , with  $f : M \rightarrow \mathbb{R}$  a smooth function. Then for any eigenvalue  $\lambda$  of the Dirac-Schrödinger operator  $D_f = D - \frac{f}{2}$ , one has*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{\frac{n}{n-1} R} - |f| \right)^2.$$

Equality holds if and only if  $M$  admits a Killing spinor and in this case  $(M^n, g)$  is Einstein,  $f$  constant, and

$$\text{sign}(\lambda) = \text{sign}(f).$$

Similarly, one has

**Proposition 7.2.** *Let  $(M^n, g)$  be a compact Riemannian spin manifold. Let  $\lambda$  be any eigenvalue of the Dirac-Schrödinger operator  $D_f = D - \frac{f}{2}$ , associated with the eigenspinor  $\varphi$ . Assume that  $R + 4|Q^\varphi|^2 > f^2 > 0$ , then*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{R + 4|Q^\varphi|^2} - |f| \right)^2.$$

where  $Q^\varphi$  the energy-momentum tensor associated with  $\varphi$ .

If equality holds,  $M$  admits an EM-spinor, and in this case,

$$\text{sign}(\lambda) = \text{sign}(f).$$

Now using a conformal change of the metric  $g$ , (see remark 5.1), we prove in the same way

**Proposition 7.3.** *Let  $(M^n, g)$  be a compact Riemannian spin manifold. Let  $\lambda$  be any eigenvalue of the Dirac-Schrödinger operator  $D_f = D - \frac{f}{2}$ , associated with the eigenspinor  $\varphi$ .*

*If  $\bar{R}e^{2u} + 4|Q^\varphi|^2 > f^2 > 0$ , where  $\bar{R}$  is the scalar curvature of  $M$  for a conformal metric  $\bar{g} = e^{2u}g$ , then*

$$\lambda^2 \geq \frac{1}{4} \inf_M \left( \sqrt{\bar{R} e^{2u} + 4|Q^\varphi|^2} - |f| \right)^2.$$

*If equality holds,  $M$  admits an WEM-spinor, and in this case, the function  $u$  is uniquely defined up to a constant by*

$$u = \frac{\ln(|\varphi|^2)}{(n-1)}.$$

Moreover,

$$\text{sign}(\lambda) = \text{sign}(f).$$

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